Noncompliance and instrumental variables for 2^K factorial experiments

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  - Can estimate the marginal, joint, & interactive effects of multiple factors.
- We also often have noncompliance.
  - ⇝GOTV phone call experiments: compliance rates of ≈20%
  - We also know what to do here with single, binary treatments—use instrumental variables approach
- But we don't know what to do with the combination of these two things!
- Almost all analyses of factorial designs assume perfect compliance….can't we do something better??

That's where we come in!

Motivation

- This paper: framework for 2^K factorial experiments with noncompliance
    1. Generalize IV assumptions and estimands to the factorial setting.
    2. Define what it can mean to be a complier in this setting.
    3. Show how to conduct finite-sample and Bayesian inference.
- Generalizes and extends previous work on 2×2 case in GOTV studies (Blackwell, 2017)
- For today, focus on randomization-based, finite-sample inference (see paper for more)

Application setting

- Blattman, Jamison, & Sheridan (2017): can noncognitive skills be changed to induce reductions in violent and criminal behavior?
- Factorial* design with two factors:
  - Cash transfers of 3 months wages
  - 8-week group cognitive behavioral therapy (CBT)

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- Setting/Sample: 999 previously violent/criminal young men in Liberia.
- Outcomes at short-term (2–5 weeks) and long-term (12–15 months):
  - Economic outcomes (income, consumption, savings)
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  - Quality of social networks (support by family, anti-sociality in network)
- Noncompliance: 2% of people didn't take cash, 37% didn't complete CBT.
- Other application in paper: 2^3 GOTV field experiment

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Refreshers

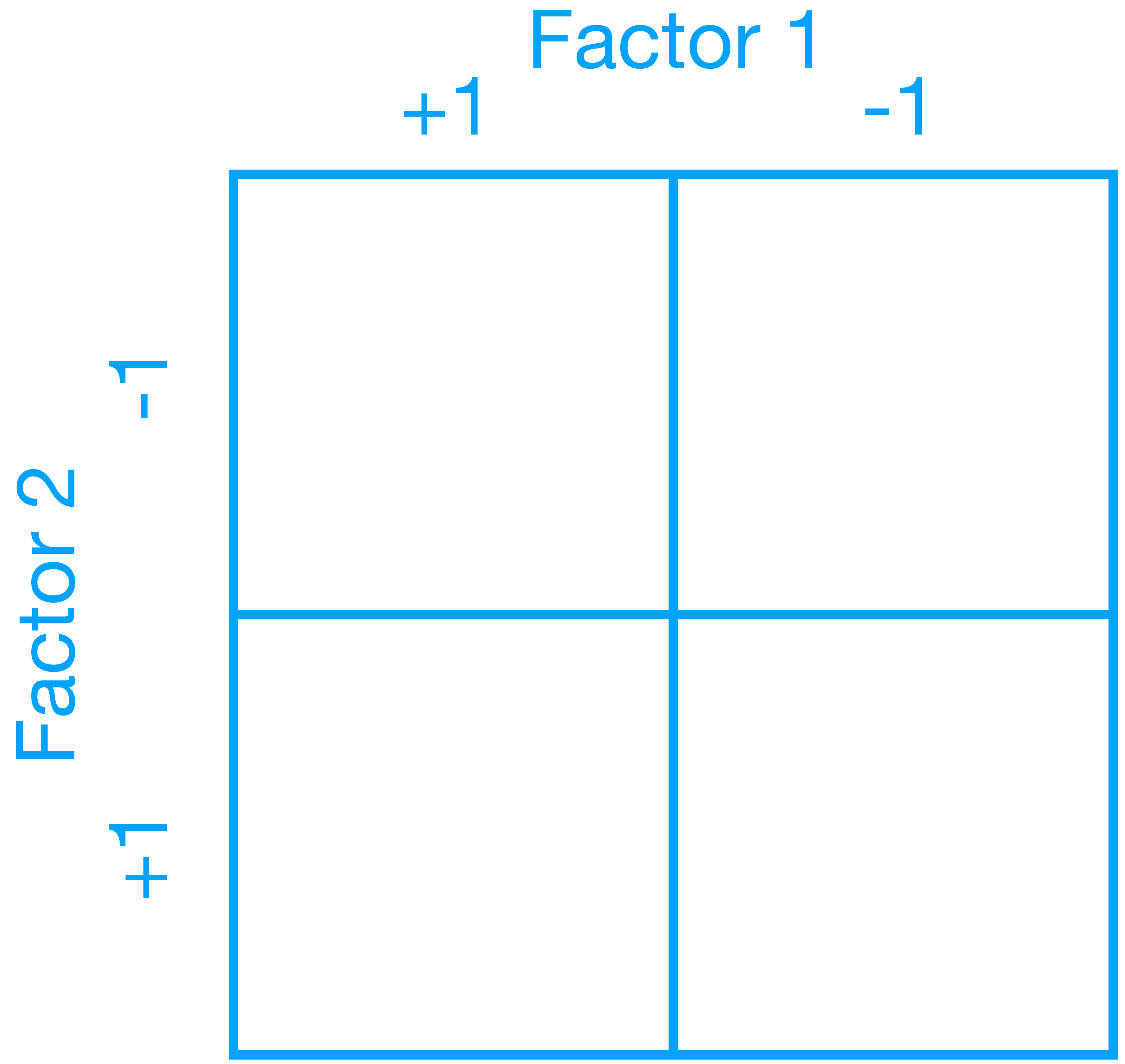
2^K factorial set up

- Have K factors, each with two levels
  - Usually label levels as +1 (treatment) and -1 (control)
- In our application, factors are
  1. Cash; levels: receive (+1) or not (-1)
  2. Therapy; levels: receive (+1) or not (-1)
- A treatment is then a combination of levels of the K factors
  - E.g., receive cash but not therapy
  - Total of 2^K possible treatments

Strange from usual potential outcomes notation, but consistent with factorial design literature

2^K factorial design

- **Design:** With N units, use complete randomization to randomly assign $N_\ell \geq 1$ units to the ℓth treatment ($\ell = 1,\ldots,2^K$, $\sum_{\ell=1}^{2^K} N_\ell = N$)



2^K factorial design

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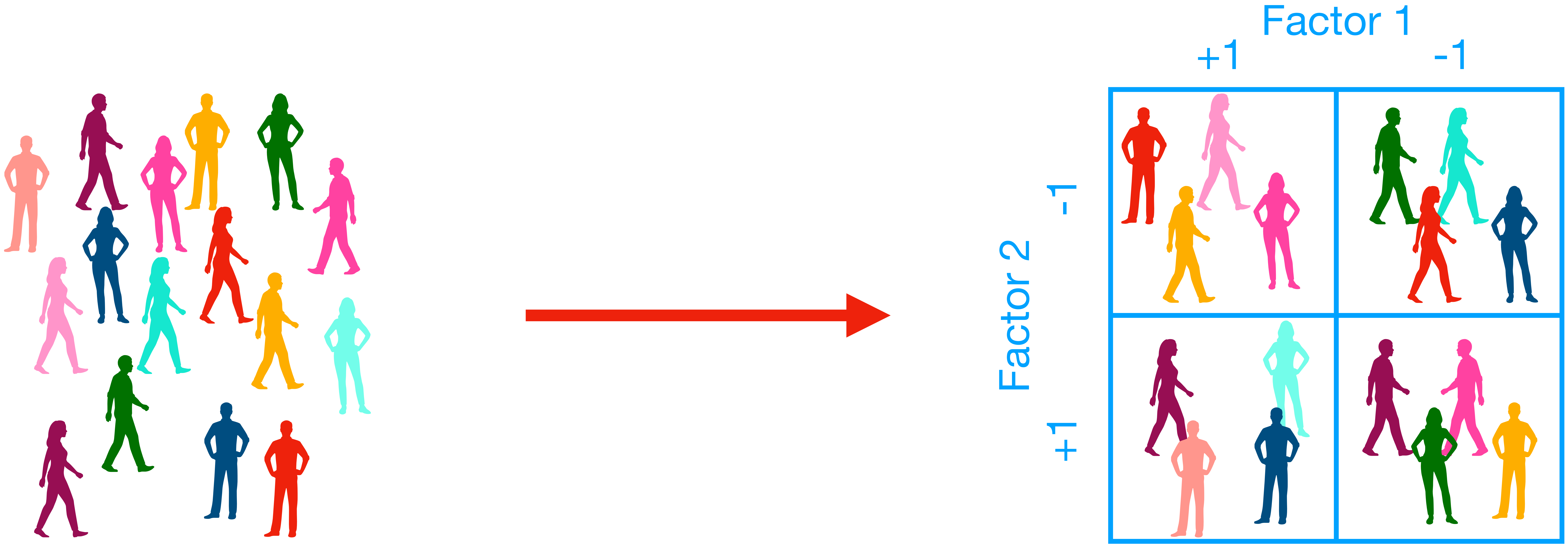
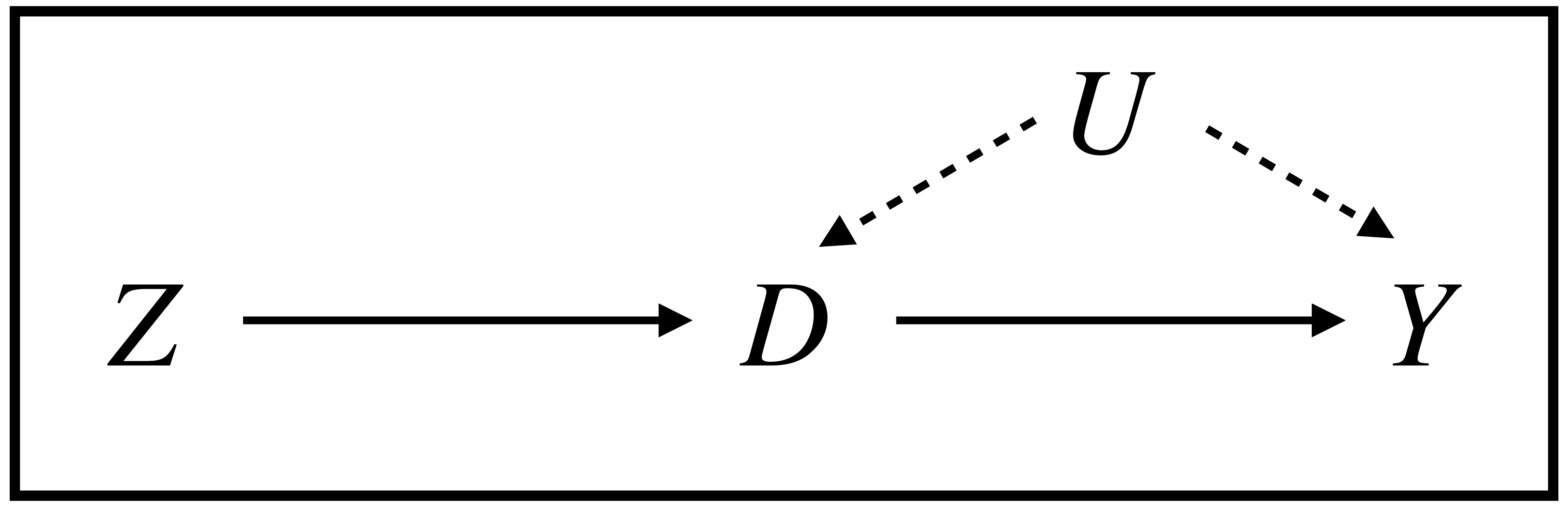


IV with a single factor



- Randomly assigned binary treatment Z_i but treatment uptake $D_i \neq Z_i$, for at least some i.
- What I really want to understand is the effect of D_i on Y_i (not randomized $\Longrightarrow$ possible confounding)

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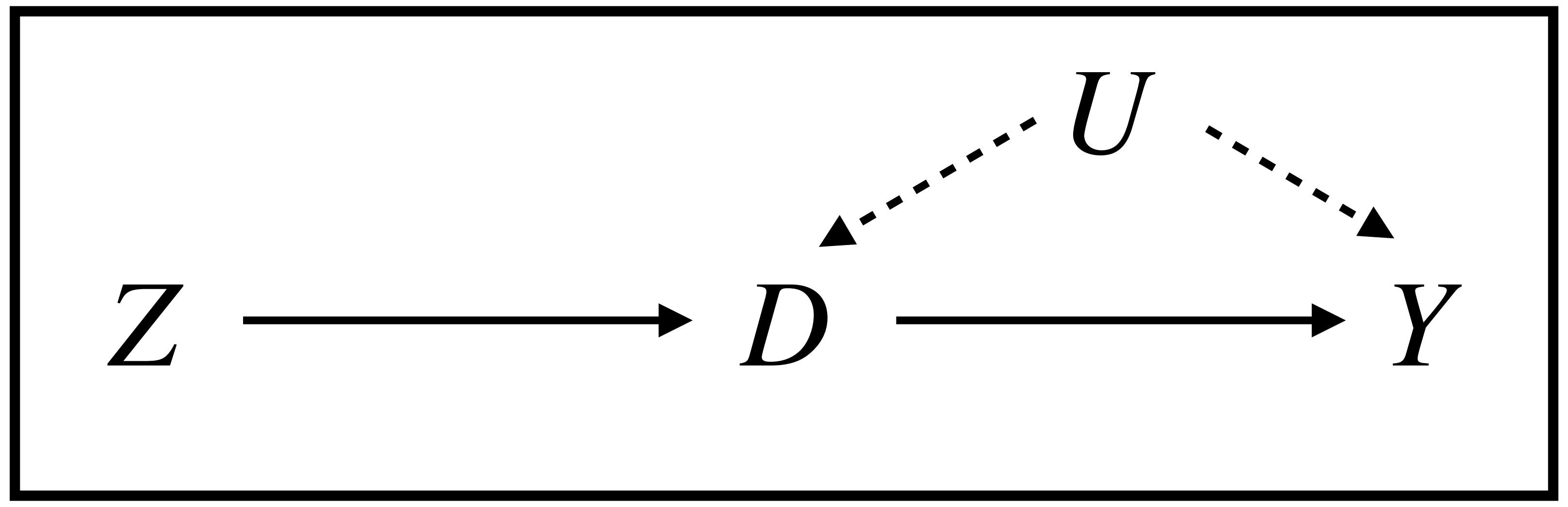

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  - $D_i(z)$: unit i's uptake for assignment z.
  - $Y_i(z, d)$: unit i's outcome under assignment z and uptake d.

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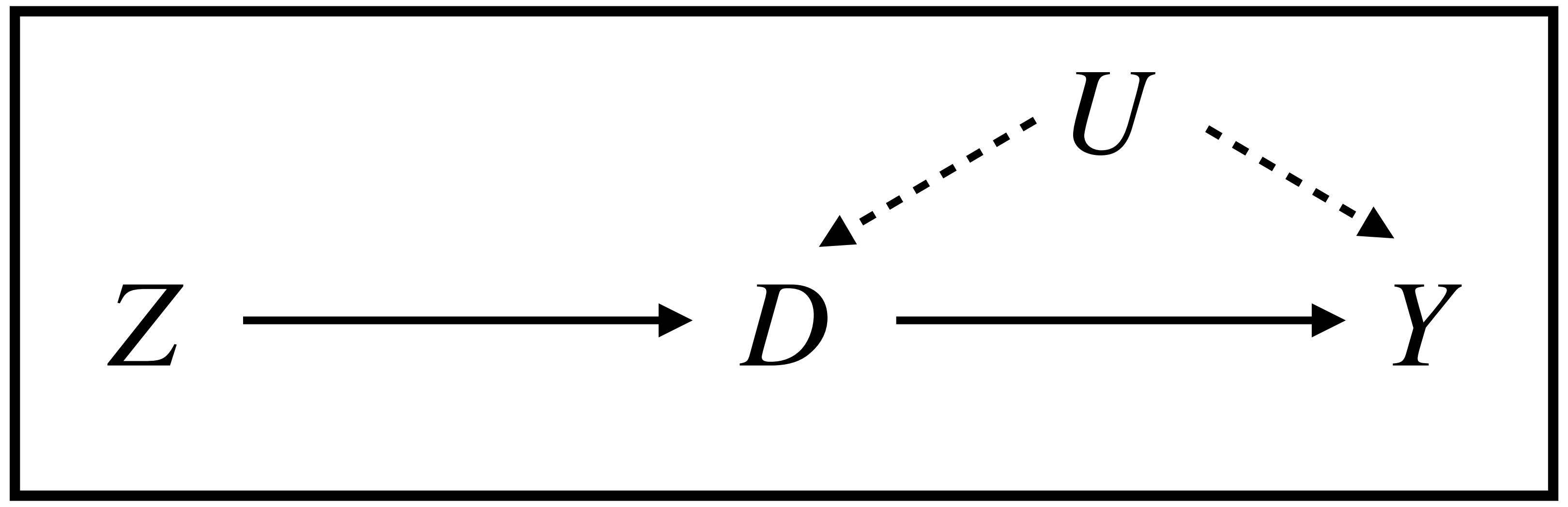

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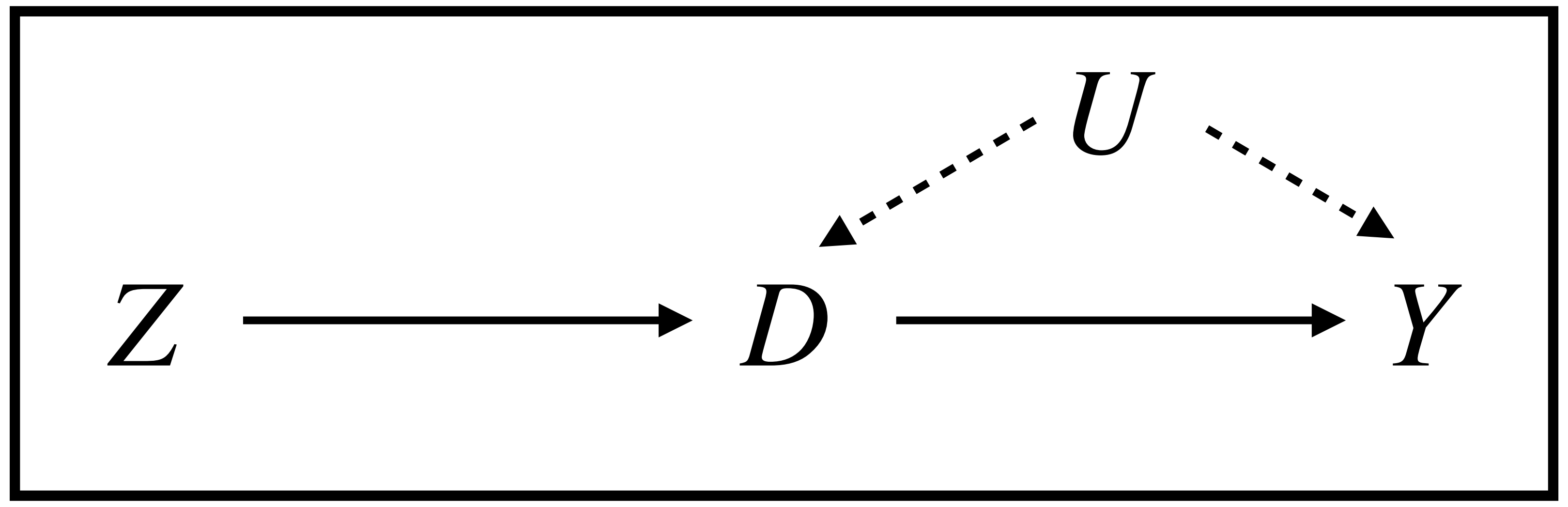
- Potential outcomes:
  - $D_i(z)$: unit i's uptake for assignment z.
  - $Y_i(z, d)$: unit i's outcome under assignment z and uptake d.
- **Principal stratification**: stratify units based on uptake profiles (i.e., on pair of $D_i(z)$ values).
  - Compliers: $D_i(z) = z$ ⟵ Do what they're told
  - Always-takers: $D_i(z) = 1$ ⟵ Take treatment no matter what
  - Never-takers: $D_i(z) = 0$ ⟵ Refuse to take treatment
  - Defiers: $D_i(z) = 1 - z$ ⟵ Do the opposite of what they are told

Can't categorize everyone from the data alone

Single-factor IV estimands



- Estimand in single factor IV is ratio of intent-to-treat (ITT) effects:

$$\tau = \frac{\mathbb{E}[Y_i \mid Z_i = 1] - \mathbb{E}[Y_i \mid Z_i = 0]}{\mathbb{E}[D_i \mid Z_i = 1] - \mathbb{E}[D_i \mid Z_i = 0]} = \frac{\text{ITT of } Z \text{ on } Y}{\text{ITT of } Z \text{ on } D}$$

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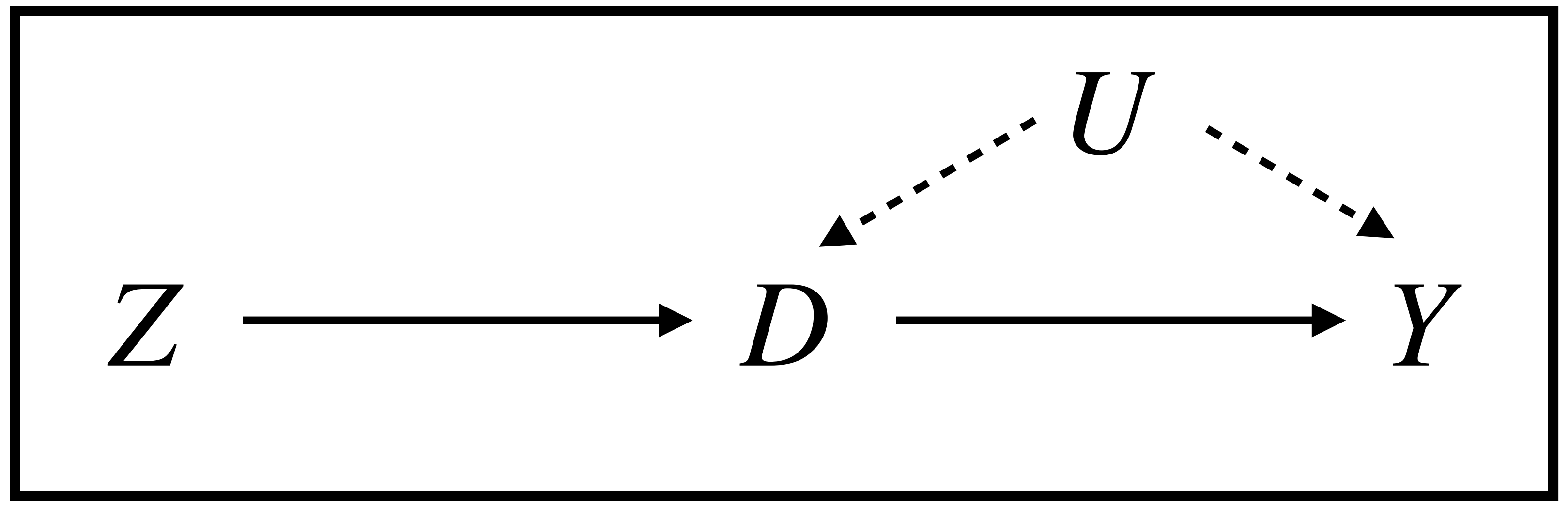

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- IV assumptions Angrist, Imbens, & Rubin (1996):
  - **Random assignment** of treatment
  - **Exclusion restriction**: $Y_i(z, d) = Y_i(z', d) = Y_i(d)$
  - **Monotonicity**: $D_i(1) \geq D_i(0) \implies$ no defiers.
  - **Nonzero effect of Z on D:** $\mathbb{E}[D_i \mid Z_i = 1] - \mathbb{E}[D_i \mid Z_i = 0] \neq 0$

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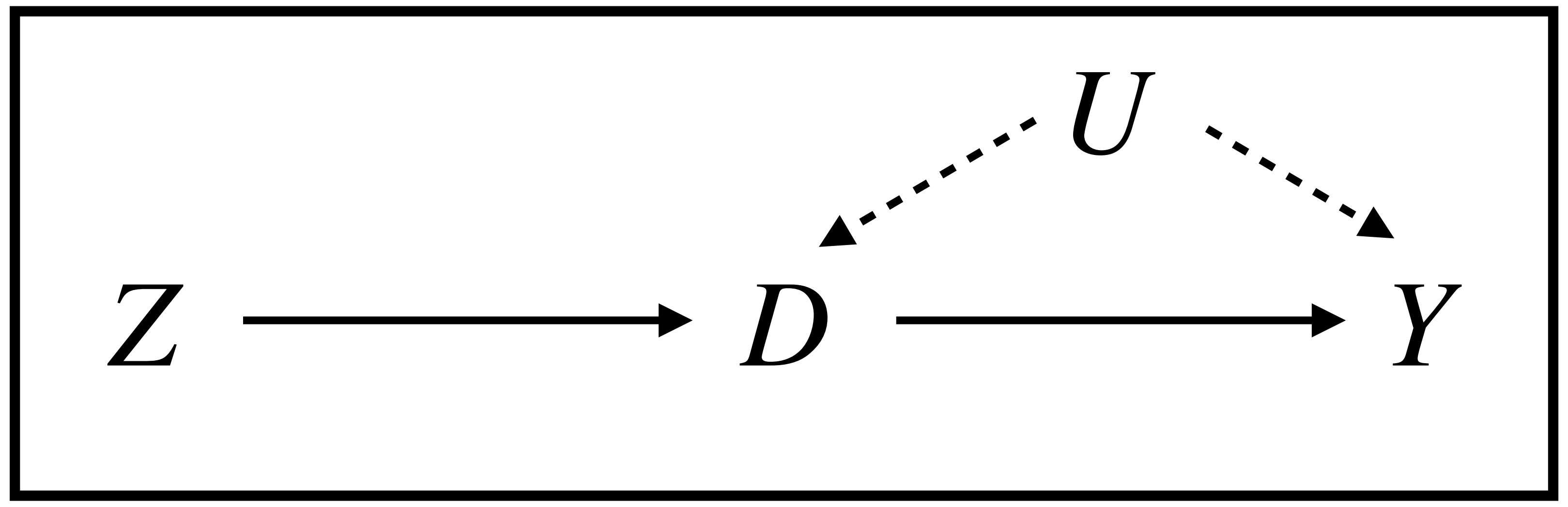

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  - **Nonzero effect of Z on D:** $\mathbb{E}[D_i \mid Z_i = 1] - \mathbb{E}[D_i \mid Z_i = 0] \neq 0$
- Under IV assumptions, τ is **local average treatment effect**, or effect among compliers.
  - An effect of D on Y even with unmeasured confounding $\rightsquigarrow$ really cool!

Our setting

Notation in the 2^K factorial setting

- Experiment with K binary factors with levels $\{-1, +1\}$
- Treatment assignments: $\mathbf{z}_\ell = \{z_{\ell 1}, \ldots, z_{\ell K}\}$

	Factor 1 ($z_{\ell 1}$)	Factor 2 ($z_{\ell 2}$)
$\mathbf{z}_1$	-1	-1
$\mathbf{z}_2$	-1	+1
$\mathbf{z}_3$	+1	-1
$\mathbf{z}_4$	+1	+1

Notation in the 2^K factorial setting

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- Treatment uptake: $\mathbf{d}_\ell = \{d_{\ell 1}, \ldots, d_{\ell K}\}$.
    - Both of these vectors have $L = 2^K$ possible values.

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  - Both of these vectors have $L = 2^K$ possible values.
- Potential treatment uptake: $\mathbf{D}_i(\mathbf{z})$
  - **Non-compliance** means $\mathbf{D}_i(\mathbf{z}) \neq \mathbf{z}$ for some i.
  - Let $D_{ik}(\mathbf{z})$ be treatment uptake for factor k given treatment assignment $\mathbf{z}$
- Potential outcomes of treatment assignment and uptake: $Y_i(\mathbf{z}, \mathbf{d})$.
  - Potential outcomes of just assignment as $Y_i(\mathbf{z}) \equiv Y_i(\mathbf{z}, \mathbf{D}_i(\mathbf{z}))$.

Vector giving levels of each factor unit i actually takes if assigned to vector of levels $\mathbf{z}$

Factorial IV assumptions

1. $\boldsymbol{Z}_i$ is completely randomized (factorial design)

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$D_{ik}(\mathbf{z})$ is level unit i actually takes for factor k if assigned treatment $\mathbf{z}$

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   - No defiers on any factor, regardless of other assignments.

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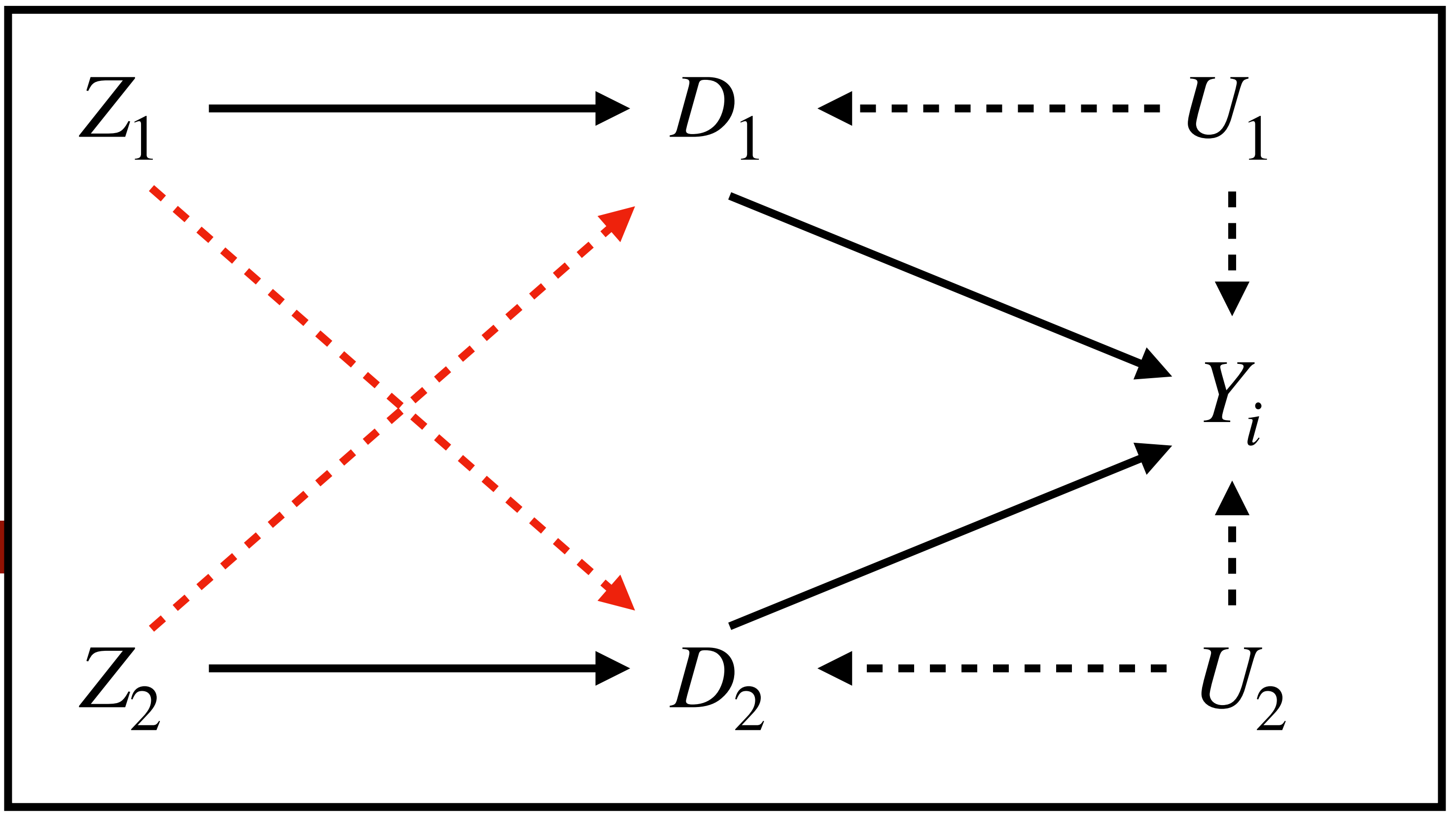
With just these assumptions, compliance status for a factor is conditional on levels of other factors $\Longrightarrow 3 \times 2^{K-1}$ possible compliance statuses for *each* factor 🤯

Factorial IV assumptions



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4. **Treatment exclusion restriction:** $D_{ik}(\mathbf{z}) = D_{ik}(z_k)$
   - Implies no effect of $\mathbf{Z}_{ik}$ on $D_{ik'}$ for $k \neq k'$. Might be implausible!
   - Blackwell (2017)

Compliance groups

- Under treatment exclusion and monotonicity, each unit can be categorized into one 3^K **compliance types** or principal strata.

- $\mathbf{T}_i = \{T_{i1}, \ldots, T_{iK}\} \in \{c, a, n\}^K$ is unit i's compliance type, where:

$$T_{ik} = \begin{cases} c & \text{if } D_{ik}(+1) = +1, D_{ik}(-1) = -1 \quad \text{(compliers)} \\ a & \text{if } D_{ik}(+1) = +1, D_{ik}(-1) = +1 \quad \text{(always takers)} \\ n & \text{if } D_{ik}(+1) = -1, D_{ik}(-1) = -1 \quad \text{(never takers)} \end{cases}$$

- Treatment exclusion is key

Estimands

Factorial effects

- IV estimand with single binary treatment is a ratio of ITTs:

$$\frac{\mathbb{E}[Y_i \mid Z_i = 1] - \mathbb{E}[Y_i \mid Z_i = 0]}{\mathbb{E}[D_i \mid Z_i = 1] - \mathbb{E}[D_i \mid Z_i = 0]} = \frac{\text{ITT of } Z \text{ on } Y}{\text{ITT of } Z \text{ on } D}$$

- **Challenge**: how to generalize this to factorial designs?

Contrast vectors

- Use the factorial effects framework of Dasgupta, Pillai, & Rubin (2015).
- Collect all potential outcomes for i into vector $\mathbf{Y}_i(\bullet)$
  - $\mathbf{Y}_i(\bullet) = \{Y_i(+1, +1), Y_i(-1, +1), Y_i(+1, -1), Y_i(-1, -1)\}$

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- Define ITT effects through contrast vectors $\mathbf{g}_j$ ($j = 1, \ldots, 2^K$):
  - L-dimensional with one half $+1$ and one half -1
  - Organize contrasts by level of interaction.
  - First K vectors, $\mathbf{g}_j$ corresponds to main (marginal) effect of factor j
  - Next $\binom{K}{2}$ vectors are two-way interactions, and so on.
- ⇝ compact definition of causal effects: $2^{-(K-1)}\mathbf{g}_j^T\mathbf{Y}_i(\bullet)$

Contrast Vectors

	Factor 1	Factor 2	$\mathbf{Y}_i(\bullet)$
$\mathbf{z}_1$	-1	-1	$Y_i(-1, -1)$
$\mathbf{z}_2$	-1	+1	$Y_i(-1, +1)$
$\mathbf{z}_3$	+1	-1	$Y_i(+1, -1)$
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$\mathbf{z}_4$	+1	+1	$Y_i(+1,+1)$
	$\mathbf{g}_1$	$\mathbf{g}_2$	

Contrast Vectors

	Factor 1	Factor 2	Interaction (1∘2)	$\mathbf{Y}_i(\bullet)$
$\mathbf{z}_1$	-1	-1	+1	$Y_i(-1, -1)$
$\mathbf{z}_2$	-1	+1	-1	$Y_i(-1, +1)$
$\mathbf{z}_3$	+1	-1	-1	$Y_i(+1, -1)$
$\mathbf{z}_4$	+1	+1	+1	$Y_i(+1, +1)$
	$\mathbf{g}_1$	$\mathbf{g}_2$		

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$\mathbf{z}_4$	+1	+1	+1	$Y_i(+1,+1)$
	$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3$	

Factorial effects example

$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3 = \mathbf{g}_1 \circ \mathbf{g}_2$	$\mathbf{Y}_i(\bullet)$
-1	-1	+1	$Y_i(-1, -1)$
-1	+1	-1	$Y_i(-1, +1)$
+1	-1	-1	$Y_i(+1, -1)$
+1	+1	+1	$Y_i(+1, +1)$

Factorial effects example: Main Effect Factor 1

$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3 = \mathbf{g}_1 \circ \mathbf{g}_2$	$\mathbf{Y}_i(\bullet)$
-1	-1	+1	$Y_i(-1,-1)$
-1	+1	-1	$Y_i(-1,+1)$
+1	-1	-1	$Y_i(+1,-1)$
+1	+1	+1	$Y_i(+1,+1)$

$$\frac{1}{2}\mathbf{g}_1^T \mathbf{Y}_i(\bullet)$$
$$= \frac{1}{2}\left(Y_i(+1,+1) - Y_i(-1,+1)\right)$$
$$+ \frac{1}{2}\left(Y_i(+1,-1) - Y_i(-1,-1)\right)$$

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-1	+1	-1	$Y_i(-1,+1)$
+1	-1	-1	$Y_i(+1,-1)$
+1	+1	+1	$Y_i(+1,+1)$

Effect of factor 1 when factor 2 is at level +1

$$\frac{1}{2}\mathbf{g}_1^T\mathbf{Y}_i(\bullet) = \frac{1}{2}\left(Y_i(+1,+1) - Y_i(-1,+1)\right) + \frac{1}{2}\left(Y_i(+1,-1) - Y_i(-1,-1)\right)$$

Effect of factor 1 when factor 2 is at level -1

Factorial effects example: Main Effect Factor 2

$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3 = \mathbf{g}_1 \circ \mathbf{g}_2$	$\mathbf{Y}_i(\bullet)$
-1	-1	+1	$Y_i(-1,-1)$
-1	+1	-1	$Y_i(-1,+1)$
+1	-1	-1	$Y_i(+1,-1)$
+1	+1	+1	$Y_i(+1,+1)$

$$\frac{1}{2}\mathbf{g}_2^T\mathbf{Y}_i(\bullet)$$

$$= \frac{1}{2}\left(Y_i(+1,+1) - Y_i(+1,-1)\right)$$

$$+ \frac{1}{2}\left(Y_i(-1,+1) - Y_i(-1,-1)\right)$$

Factorial effects example: Main Effect Factor 2

$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3 = \mathbf{g}_1 \circ \mathbf{g}_2$	$\mathbf{Y}_i(\bullet)$
-1	-1	+1	$Y_i(-1,-1)$
-1	+1	-1	$Y_i(-1,+1)$
+1	-1	-1	$Y_i(+1,-1)$
+1	+1	+1	$Y_i(+1,+1)$

Effect of factor 2 when factor 1 is at level +1

$$\frac{1}{2}\mathbf{g}_2^T\mathbf{Y}_i(\bullet)$$

$$= \frac{1}{2}\left(Y_i(+1,+1) - Y_i(+1,-1)\right)$$

$$+ \frac{1}{2}\left(Y_i(-1,+1) - Y_i(-1,-1)\right)$$

Effect of factor 2 when factor 1 is at level -1

Factorial effects example: Interaction

$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3 = \mathbf{g}_1 \circ \mathbf{g}_2$	$\mathbf{Y}_i(\bullet)$
-1	-1	+1	$Y_i(-1,-1)$
-1	+1	-1	$Y_i(-1,+1)$
+1	-1	-1	$Y_i(+1,-1)$
+1	+1	+1	$Y_i(+1,+1)$

$$\frac{1}{2}\mathbf{g}_3^T\mathbf{Y}_i(\bullet)$$

$$= \frac{1}{2}\left(Y_i(+1,+1) - Y_i(-1,+1)\right)$$

$$-\frac{1}{2}\left(Y_i(+1,-1) - Y_i(-1,-1)\right)$$

Factorial effects example: Interaction

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-1	+1	-1	$Y_i(-1, +1)$
+1	-1	-1	$Y_i(+1, -1)$
+1	+1	+1	$Y_i(+1, +1)$

Effect of factor 1 when factor 2 is at level +1

$$\frac{1}{2}\mathbf{g}_3^T \mathbf{Y}_i(\bullet)$$

$$= \frac{1}{2}\left(Y_i(+1, +1) - Y_i(-1, +1)\right)$$

$$-\frac{1}{2}\left(Y_i(+1, -1) - Y_i(-1, -1)\right)$$

Effect of factor 1 when factor 2 is at level -1

Intent-to-treat effects on the outcome

- Finite-sample (aka finite-population) average potential outcomes: $\overline{\mathbf{Y}}(\bullet) = N^{-1} \sum_{i=1}^{N} \mathbf{Y}_i(\bullet)$.
- Finite-sample average ITT factorial effect:

$$\bar{\tau}_j = 2^{-(K-1)} \mathbf{g}_j^T \overline{\mathbf{Y}}(\bullet)$$

Intent-to-treat effects on the outcome

- Population average potential outcomes: $\overline{\mathbf{Y}}(\bullet) = N^{-1} \sum_{i=1}^{N} \mathbf{Y}_i(\bullet)$.
- Finite-sample average ITT factorial effect:

$$\bar{\tau}_j = 2^{-(K-1)} \mathbf{g}_j^T \overline{\mathbf{Y}}(\bullet)$$

- Interpretations of the $\bar{\tau}_j$ as different effect of assignment:
  - $j = 1, \ldots, K$: main effect of factor j, marginalizing over other factors.
  - $j > K$: interaction among some factors, marginalizing over the rest.
  - Similar to average marginal component effects in conjoint designs.
- These are effects **in the sample** not in some hypothetical population.

Intent-to-treat effects on treatment uptake

- What does an ITT for treatment uptake mean in this setting?

Intent-to-treat effects on treatment uptake

- What does an ITT for treatment uptake mean in this setting?
- Consider how we defined the ITT effects on the outcome
- Let $\mathbf{D}_{ik}(\bullet)$ be the vector of $D_{ik}(\mathbf{z})$ for all $\mathbf{z}$
- For main effects, can use this to get ITT:
  - e.g., $\bar{\delta}_1 = 2^{-(K-1)}\mathbf{g}_1^T\mathbf{D}_{i1}(\bullet)$

Intent-to-treat effects on treatment uptake

- What does an ITT for treatment uptake mean in this setting?
- Consider how we defined the ITT effects on the outcome
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- For main effects, can use this to get ITT:
  - e.g., $\bar{\delta}_1 = 2^{-(K-1)}\mathbf{g}_1^T\mathbf{D}_{i1}(\,\bullet\,)$
- But it's more complicated for interactions…we'll need new notation

Intent-to-treat effects on treatment uptake

- **Active factors** for effect j: factors being contrasted in effect j. For 2×2:
  - $j = 1 \rightsquigarrow$ main effect of factor 1 so active factor is just $\{1\}$.
  - $j = 3 \rightsquigarrow$ interaction between factor 1 and factor 2 so active factors are $\{1,2\}$

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- $\widetilde{\mathbf{D}}_{ij}$: The **element-wise products/interactions** of uptake between active factors for jth factorial effect
  - $\widetilde{\mathbf{D}}_{i1}(\bullet) = \mathbf{D}_{i1}(\bullet)$ and $\widetilde{\mathbf{D}}_{i3}(\bullet) = \mathbf{D}_{i1}(\bullet) \circ \mathbf{D}_{i2}(\bullet)$

Like how we defined $\mathbf{g}_3$

For factor 1, treatment uptake for factor 1

Interaction, multiply treatment uptake for factor 1 and factor 2

Intent-to-treat effects on treatment uptake

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  - $\widetilde{\mathbf{D}}_{i1}(\bullet) = \mathbf{D}_{i1}(\bullet)$ and $\widetilde{\mathbf{D}}_{i3}(\bullet) = \mathbf{D}_{i1}(\bullet) \circ \mathbf{D}_{i2}(\bullet)$
- Treatment uptake ITT, $\bar{\delta}_j$, are the jth factorial effect on $\widetilde{D}_{ij}$
  - $\bar{\delta}_1 = 2^{-(K-1)}\mathbf{g}_1^T\widetilde{\mathbf{D}}_{i1}(\bullet) \rightsquigarrow$ main effect of Z_{i1} on D_{i1}.
  - $\bar{\delta}_3 = 2^{-(K-1)}\mathbf{g}_3^T\widetilde{\mathbf{D}}_{i3}(\bullet) \rightsquigarrow$ interaction effect between Z_{i1} and Z_{i2} on $D_{i1}D_{i2}$.

Marginalized-complier average factorial effects (MCAFEs)

- IV ratio estimands for the factorial effects:

$$\overline{\phi}_j = \frac{\overline{\tau}_j}{\overline{\delta}_j} = \frac{j\text{th factorial ITT effect on } Y}{j\text{th factorial ITT effect on } \widetilde{D}_j}$$

- Without IV assumptions, $\overline{\phi}_j$ has the interpretation as a ratio of effects.
- With factorial IV assumptions:
    - $\overline{\delta}_j$: proportion of compliers with the active factors for effect j
    - $\overline{\phi}_j$: **marginalized-complier average factorial effect** or MCAFE.
    - $\Longrightarrow$ average factorial effect among compliers with the active factors.
- Also need to assume $\overline{\delta}_j \neq 0$ (some who comply for all active factors in j)

Perfect compliers

- Problem with MCAFE: conditioning set changes across factorial effects.
  - ⇝difficult to compare, e.g., main effects and interactions.
- Example: interactions between factors 1 and 2 in 2^3 design.
  - Marginalized compliers: $\mathbf{T}_i$ is either $\{c, c, a\}$, $\{c, c, n\}$, or $\{c, c, c\}$

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  - Marginalized compliers: $\mathbf{T}_i$ is either $\{c, c, a\}$, $\{c, c, n\}$, or $\{c, c, c\}$
- Alternative: effects among compliers on **all** factors (**perfect compliers**)
  - Perfect compliers: $\mathbf{T}_i = \{c, c, c\}$
  - Define $\bar{\gamma}_j$ as the jth **perfect complier average factorial effect** (PCAFE).
  - Also can write as a (more complicated) ratio of effects.

Inference

Observed data

- Treatment indicators: $W_{i\ell} = 1$ if $\mathbf{Z}_i = \mathbf{z}_\ell$.
- Number of units assigned to each treatment vector: $N_\ell = \sum_{i=1}^{N} W_{i\ell}$
- Average observed potential outcomes: $\widehat{Y}_i(\mathbf{z}_\ell) = N_\ell^{-1} \sum_{i=1}^{N} W_{i\ell} Y_i(\mathbf{z}_\ell)$

Estimators

- Estimators for the ITT effects: Estimate $\hat{\tau}_j$ by replacing $\overline{Y}(\mathbf{z})$ with $\widehat{Y}(\mathbf{z})$

$$\hat{\tau}_1 = \frac{1}{2}\{\widehat{Y}(+1, +1) - \widehat{Y}(-1, +1)\} + \frac{1}{2}\{\widehat{Y}(+1, -1) - \widehat{Y}(-1, -1)\}$$

  - Similar estimator $\hat{\delta}_j$ for ITT effects on treatment uptake, replace Y with D.

- Randomization $\Longrightarrow$ $\hat{\tau}_j$ and $\hat{\delta}_j$ unbiased for the ITT effects $\overline{\tau}_j$ and $\overline{\delta}_j$

- Ratio estimators for the MCAFEs (similar for the PCAFEs):

$$\widehat{\phi}_j = \hat{\tau}_j / \hat{\delta}_j$$

Finite-sample asymptotics

- We focus on **finite-sample inference**:
  - View potential outcomes as fixed (focus on units in experiment).
  - Randomization of treatments is the only source of variation (no sampling).
  - No assumptions about the data generating process of the outcome.

Finite-sample asymptotics

- We focus on **finite-sample inference**:
  - View potential outcomes as fixed (focus on units in experiment).
  - Randomization of treatments is the only source of variation (no sampling).
  - No assumptions about the data generating process of the outcome.
- We will use **finite-population central limit theorems** (allows weak nulls)
  - Normal approximations as finite-population sample sizes grow.
  - ⇝ the joint distribution of the ITT estimators $\widehat{\tau}_j$ and $\widehat{\delta}_j$ are **approximately normal** in large (finite) samples. (Li & Ding, 2017)

Inference for ratio estimators

- Two broad approaches to constructing confidence intervals for these ratio estimators $\widehat{\phi}_j = \hat{\tau}_j / \hat{\delta}_j$

1. **Delta method**
   - Use delta method (combine Li & Ding (2017) and Pashley (2019)) to get $V_j = \mathbb{V}[\hat{\tau}_j / \hat{\delta}_j]$ and derive estimator
   - Construct 95% CIs via $\widehat{\phi}_j \pm 1.96 \times \widehat{V}_j^{1/2}$
   - Poor approximation with few compliers (denominator is ≈ 0)
2. **Fieller's method (**Fieller, 1954)
   - Analytically invert test statistic to construct CIs using quadratic formula.
   - Confidence intervals have excellent coverage even with low compliance rates…
   - ...but CIs can have infinite length (accurately reflects uncertainty?)
   - Used by Kang, Peck, & Keele (2018) and Li & Ding (2017) for univariate IV setting.

Application: Effects of Cash and Cognitive Behavioral Therapy on Crime

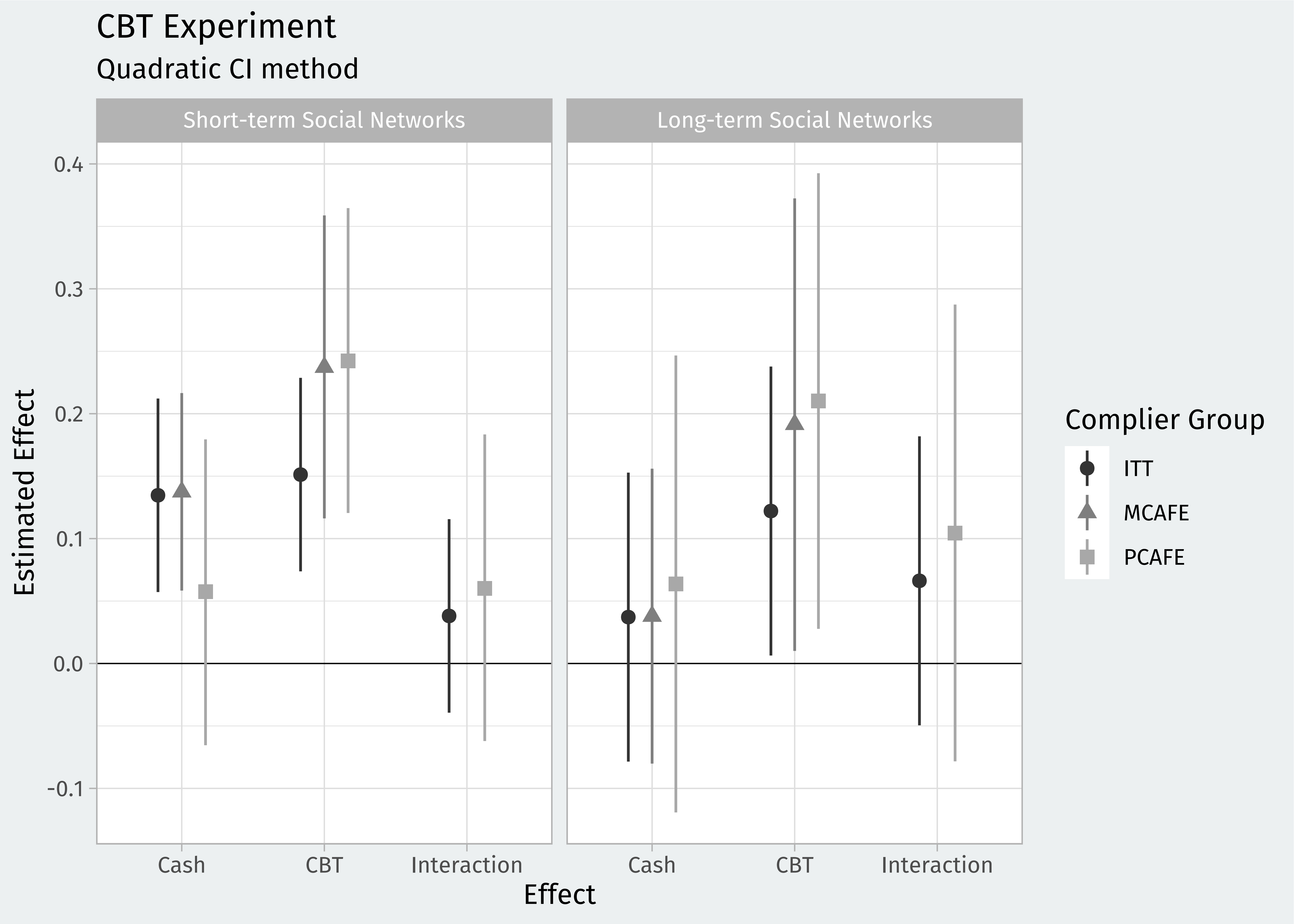
Compliance

- Marginal compliance rate with cash transfers: 0.98
- Marginal compliance rate with CBT treatment: 0.63
  - Compliance defined as attended > 80% of CBT meetings.
- Perfect compliers are roughly compliers for CBT
  - ⇝PCAFEs and MCAFEs should be similar for CBT.
  - Might differ for cash treatment.

	Assigned CBT	Assigned No CBT
Assigned Cash	249	250
Assigned No Cash	280	220

	Attended CBT	No CBT
Received Cash	161	328
No Cash	176	334

Results: Quality of social networks



Discussion

- Social science experiments often have:
  1. multiple factors
  2. noncompliance on at least one of these factors.
- This paper provides a framework to handle both features.
- Contributions:
  - Generalize the IV assumptions and estimands to the factorial setting.
  - Differentiate different marginalized and perfect compliers.
  - Consistent estimators for MCAFEs and PCAFEs.
  - Provide confidence interval procedures valid for finite-sample inference.
  - Implementation available in the `factiv` R package.

Discussion

- What next?
  - Weakening the treatment exclusion restriction (bounds analysis).
  - Allow for more than 2 levels per factor.
  - Using covariates to characterize marginal compliers
  - Handling multiple comparisons in this setting.

Thanks!

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